A weaker notion of the finite factorization property

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Outline

General Notation

General notation we will use throughout this talk:

- $\mathbb{N} := \{1, 2, 3, \ldots\}$,
- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$,
- $\mathbb{P}$ denotes the set of primes,
- $3 = p_1 < p_2 < \cdots$ is the sequence of odd primes.

A Convention and Some Preliminaries

A monoid is a cancellative commutative semigroup with an identity element.

Definitions: Let M be a (additive) monoid.

- We let $\mathcal{U}(M)$ denote the group of units (i.e., invertible elements) of M.
- We let $M^\bullet$ denote the set of nonzero elements of M.
- A submonoid of M is a subset of M that is closed under addition and contains 0.
- A positive monoid is an additive submonoid of $\mathbb{R}_{\geq 0}$.
- If S is a subset of M, then $\langle S \rangle$ stands for the smallest submonoid of M that contains S.
- $a \in M \setminus \mathcal{U}(M)$ is an atom (or an irreducible) if for any $b, c \in M$ the equality $a = b + c$ implies that either $b \in \mathcal{U}(M)$ or $c \in \mathcal{U}(M)$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- An element of M is atomic if it is a unit or it factors into atoms.
- A subset I of M is an ideal if $I + M := \{bm \mid b \in I \text{ and } m \in M\} \subseteq I$.

Atomicity and the ACCP

Definition

A monoid is called atomic if each nonzero nonunit factors into atoms (every element of M is atomic).

- An ideal I of M is called principal if $I = b + M$ for some $b \in M$.
- A sequence of ideals $(I_n)_{n \geq 1}$ of M is called an ascending chain if $I_n \subseteq I_{n+1}$ for every $n \in \mathbb{N}$.

Definition

The monoid M is said to satisfy the ACCP if every ascending chain of principal ideals $(I_n)_{n \geq 1}$ stabilizes.

Remark: Atomicity and the ACCP has been well studied.

- If a monoid satisfies the ACCP, then it is atomic.
- The converse does not hold. Consider the Grams' monoid $M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle$.

Bounded and Finite Factorization Properties

- We let $\mathsf{Z}(M)$ denote the monoid consisting of all formal sums of atoms in M. The elements of $\mathsf{Z}(M)$ are called factorizations in M.
- For each $b \in M$, we let $\mathsf{Z}(b)$ denote all formal sums of atoms that sum to b.
- If $z := a_1 + \cdots + a_\ell \in \mathsf{Z}(b)$ for some $a_1, \ldots, a_\ell \in \mathcal{A}(M)$, then we call z the factorization of b and ℓ the length of z, and we often denote ℓ by $|z|$.
- For $b \in M$, the set $\mathsf{L}(b) := \{|z| \mid z \in \mathsf{Z}(b)\}$ is called the set of lengths of b.

Definition

The monoid M is called an FFM if $1 \le |\mathsf{Z}(b)| < \infty$ for every $b \in M$.

Definition

We say that M is a BFM if $1 \le |\mathsf{L}(b)| < \infty$ for every $b \in M$.

Remarks:

- It follows from the definitions that every FFM is a BFM.
- It is well known that every BFM satisfies the ACCP.
- The implication chain would be $\mathit{FFM} \Rightarrow \mathit{BFM} \Rightarrow \mathit{ACCP} \Rightarrow \mathit{Atomic}$.

Length-Finite Factorization Property (LFFM)

For each $b \in M$ and $\ell \in \mathbb{N}$, set $\mathsf{Z}_\ell(b) := \{z \in \mathsf{Z}(b) \mid |z| = \ell\}$.

Definition (Geroldinger-Zhong, 2021)

We say that M is a length-finite factorization monoid (LFFM) if M is atomic and $|\mathsf{Z}_\ell(b)| < \infty$ for all $b \in M$ and $\ell \in \mathbb{N}$.

From the definitions, we obtain that every FFM is an LFFM.

Questions:

- Can we identify a large class of LFFMs?
- How does LFFM compare to other well-studied properties such as BFM and ACCP?

Co-well-ordered Monoids

- We say that a positive monoid is decreasing (resp., increasing) if it can be generated by a decreasing (resp., increasing) sequence.
- We say that a sequence of real numbers is well-ordered (resp., co-well-ordered) if it contains no strictly decreasing (resp., increasing) subsequence.

Definition (Harold Polo, 2022)

We say that a positive monoid M is well-ordered (resp., co-well-ordered) if it can be generated by a well-ordered (resp., co-well-ordered) sequence.

Remarks:

- A positive monoid is both decreasing and increasing if and only if it is finitely generated.
- It follows from the definitions that every increasing (resp., decreasing) positive monoid is well-ordered (resp., co-well-ordered).

A Class of LFFMs

Theorem (LFFM Theorem, Jiang-Kanungo-Kim, 2023)

Every atomic co-well-ordered positive monoid is an LFFM.

Before we prove this theorem, there are a few lemmas that will help.

Lemma (Jiang-Kanungo-Kim, 2023)

The sum of finitely many co-well-ordered sequences is a co-well-ordered sequence.

- A pair (z_1, z_2) of factorizations in $\mathsf{Z}(M)$ of the same element of M is called irredundant if z_1 and z_2 do not share any atoms.
- We say that a subset Z of $\mathsf{Z}(M)$ is irredundant if any two distinct factorizations in Z are irredundant.

Lemma (Jiang-Kanungo-Kim, 2023)

Let M be an atomic co-well-ordered positive monoid. For all $x \in M$ and $\ell \in \mathbb{N}$, every irredundant subset of $\mathsf{Z}_\ell(x)$ is finite.

Outline of Proof of Theorem

- Let M be an atomic co-well-ordered positive monoid. Suppose that M is not an LFFM.
- Then for some $\ell \in \mathbb{N}$ there exists $x \in M$ such that $|\mathsf{Z}_\ell(x)| = \infty$. Assume that we have chosen ℓ as the smallest as it can possibly be.

Lemma (Zorn's Lemma)

If X is a partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.

- There exists a maximal irredundant subset of $\mathsf{Z}_\ell(x)$ under inclusion by Zorn's lemma.
- Let Z be the maximal irredundant subset and let A be the set consisting of all the atoms of M that appear in at least one factorization in Z.
- The maximality of Z guarantees that every factorization in $\mathsf{Z}_\ell(x)$ contains an atom from A.
- Set $Z_a := \{z \in \mathsf{Z}_\ell(x) \mid a \text{ appears in } z\}$ for each $a \in A$. Since A is finite, the fact that $\mathsf{Z}_\ell(x)$ is infinite ensures the existence of $a \in A$ such that $|Z_a| = \infty$.

Outline of Proof of Theorem

- The set of factorizations $Z'_a := \{z - a \mid z \in Z_a\}$ is a subset of $\mathsf{Z}_{\ell-1}(x - a)$.
- Since $|Z'_a| = |Z_a| = \infty$, it follows that $\mathsf{Z}_{\ell-1}(x - a)$ is infinite, which contradicts the minimality of ℓ.
- Hence M is an LFFM.

Comparing LFFM to BFM and ACCP

Example of a BFM that is not an LFFM:

- Consider the positive monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$.
- We can verify that $\mathcal{A}(M) = \mathbb{Q} \cap [1, 2)$ and, therefore, that M is atomic.
- Since 0 is not a limit point of $M^\bullet$, we know that M is a BFM.
- The equality $3 = \left(\frac{3}{2} - \frac{1}{n}\right) + \left(\frac{3}{2} + \frac{1}{n}\right)$ yields a length-2 factorization of 3 for each $n \in \mathbb{N}$ with $n \geq 3$. Thus M is not an LFFM.

Example of an LFFM that does not satisfy the ACCP:

- Consider the Grams' monoid: $M := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$.
- It is well known that M is atomic with $\mathcal{A}(M) = \left\{ \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\}$ but does not satisfy the ACCP since $(\frac{1}{2^n} + M)_{n \geq 0}$ is an ascending chain of principal ideals that does not stabilize.
- Since the sequence of defining generators of M is decreasing, M is a decreasing monoid and, therefore, M is an LFFM by our theorem.

Comparing LFFM to BFM and ACCP

Thus, we can conclude that BFM and LFFM properties are **not comparable** and LFFM and ACCP are also **not comparable**.

Remarks: We can form two separate implication chains.

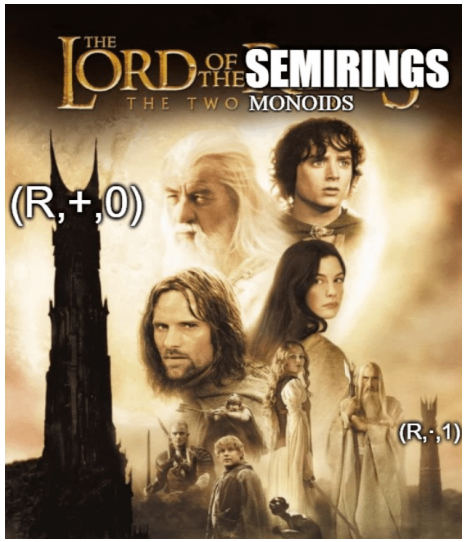
- $FFM \Rightarrow BFM \Rightarrow ACCP \Rightarrow Atomic$
- $FFM \Rightarrow LFFM \Rightarrow Atomic$

Semirings

A semiring is a triple $(S, +, \cdot)$ of a set and two binary operations such that

- $(S, +)$ is a monoid with identity denoted by 0.
- $(S \setminus \{0\}, \cdot)$ is a monoid with identity denoted by 1.
- $+$ and $\cdot$ distribute, i.e.

$$a(b + c) = ab + ac$$



When the operations are clear, we will write S instead of $(S, +, \cdot)$. We say that the monoid $(S \setminus \{0\}, \cdot)$ is the multiplicative monoid of S, denoted by $S^\bullet$.

- S is a BFS (resp. LFFS, FFS) if its multplicative monoid $S^\bullet$ is a BFM (resp. LFFM, FFM).

Ex: $(\mathbb{N}_0, +, \cdot)$, $(\mathbb{Q}_{\geq 0}, +, \cdot)$, and $(\mathbb{R}_{\geq 0}, +, \cdot)$ are all semirings.

Subsemirings

Suppose $(S, +, \cdot)$ is a semiring.

Definition

If H is a subset of S such that H contains 0 and 1 and is closed under addition and multiplication, we say H is a subsemiring of S. Note that H is also a semiring.

A subsemiring of $(\mathbb{R}_{\geq 0}, +, \cdot)$ is a positive semiring. Note that the elements of a positive semiring form a positive monoid under addition.

Ex: $(\mathbb{N}_0, +, \cdot)$ is a subsemiring of $(\mathbb{Q}_{\geq 0}, +, \cdot)$ which is a subsemiring of $(\mathbb{R}_{\geq 0}, +, \cdot)$, and they are all positive semirings.

We are lazy, so we don't want to come up with brand new positive semirings to investigate the ACCP, BFS, LFFS, and FFS properties. So what do we do?

We turn positive monoids into positive semirings!

The $E(M)$ Construction

Set M to be a positive monoid whose elements are all algebraic numbers.

Definition

Let $E(M)$ be the positive monoid generated by $\{e^m \mid m \in M\}$.

- $E(M)$ contains 1 and $E(M)$ is closed under multiplication. Thus, $E(M)$ is a positive semiring.
- Since the elements of M are algebraic numbers, by the *Lindemann-Weierstrass Theorem*, every element of $E(M)$ can be written uniquely as

$$r = c_1 e^{m_1} + \cdots + c_k e^{m_k}$$

  where $m_1 < \cdots < m_k$ and the c_i's are positive integers.
- Define $\mathrm{LC}(r) := c_k$ and $\deg r := m_k$ the leading coefficient and the degree of r, respectively. In addition, we call $S_E(r) := \{m_1, \ldots, m_k\}$ the exponent set of r.
- If $a_1, \ldots, a_k \in E(M)$ and $m_i \in S_E(a_i)$, we can show that

$$\deg\left(\prod_{i=1}^{k} a_i\right) = \sum_{j=1}^{k} \deg a_j \text{ and } \mathrm{LC}\left(\prod_{i=1}^{n} a_i\right) = \prod_{j=1}^{n} \mathrm{LC}(a_j) \text{ and } \sum_{i=0}^{k} m_i \in S_E\left(\prod_{i=1}^{k} a_i\right)$$

The ACCP Property and $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid satisfying the ACCP, then $E(M)$ also satisfies the ACCP.

Sketch of proof:

- Suppose to the contrary that $E(M)$ does not satisfy the ACCP.
- Then there exists an element $s \in E(M)$ and two sequences $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$ whose terms are nonunits of $E(M)$ such that $s = t_n s_1 s_2 \cdots s_n$ for every $n \in \mathbb{N}$.
- Then

$$\deg s = \deg t_n + \sum_{i=1}^{n} \deg s_i \quad \text{and} \quad \mathsf{LC}(s) = \mathsf{LC}(t_n) \cdot \prod_{i=1}^{n} \mathsf{LC}(s_i).$$

- Let ℓ be the length of the factorization of $\mathsf{LC}(s)$ in $\mathbb{N}$.

The ACCP Property in $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid satisfying the ACCP, then $E(M)$ also satisfies the ACCP.

- At most ℓ of the real numbers $\mathrm{LC}(s_1), \ldots, \mathrm{LC}(s_n), \mathrm{LC}(t_n)$ can be greater than one.
- For $q \in E(M)$, if $LC(q) = 1$ and $\deg q = 0$, then $q = 1$ (so q is a unit).
- Since s_k and t_k are nonunits for every $k \in \mathbb{N}$, at least $n - \ell$ of the real numbers $\deg s_1, \ldots, \deg s_n$ must be nonzero, so infinitely many of $\deg s_1, \deg s_2, \ldots$ must be nonzero.
- Then the ascending chain of principal ideals $\deg t_1 + M, \deg t_2 + M, \ldots$ in M does not stabilize because $\deg t_n = \deg t_{n-1} + \deg s_n$.

The BFS Property in $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid that is a BFM, then $E(M)$ is a BFS.

Sketch of proof:

- M satisfies the ACCP, so $E(M)$ satisfies the ACCP, so $E(M)$ is atomic.
- Suppose to the contrary that $E(M)$ is not a BFS. Then there exists $q \in E(M)$ such that there is no bound to the lengths of factorizations of q.
- $q = q_1 \cdots q_n$ for nonunits $q_1, \ldots, q_n$.
- Then

$$\deg q = \sum_{i=1}^{n} \deg q_i \quad \text{and} \quad \mathsf{LC}(q) = \prod_{i=1}^{n} \mathsf{LC}(q_i).$$

- Let ℓ be the length of the factorization of $\mathsf{LC}(q)$ in $\mathbb{N}$.

The BFS Property in $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid that is a BFM, then $E(M)$ is a BFS.

- At most ℓ of the real numbers $\mathrm{LC}(q_1), \ldots, \mathrm{LC}(q_n)$ can be greater than one.
- For $r \in E(M)$, if $LC(r) = 1$ and $\deg r = 0$, then $r = 1$ (so r is a unit).
- Since q_k is a nonunit for every $k \in \mathbb{N}$, at least $n - \ell$ of the real numbers $\deg q_1, \ldots, \deg q_n$ must be nonzero.
- Since $\deg q = \deg q_1 + \cdots \deg q_n$, this means q has a factorization in M with at least $n - \ell$ terms.
- But n can be arbirarily large, so there is no bound to the lengths of factorizations of q in M, so M is not a BFS, a contradiction. Thus $E(M)$ is a BFS.

The FFS Property in $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid that satisfies the FFM property, then $E(M)$ is a FFS.

Sketch of proof:

- Suppose M is a FFM. Then $E(M)$ must be a BFS.
- We will show that for any positive integer ℓ and $q \in E(M)$, there are finitely many ways to choose atoms $a_1, \ldots, a_\ell$ such that $q = a_1 \cdots a_\ell$.
- Set $m := m_1 + \cdots + m_\ell$, where $m_1 \in S_E(a_1), \ldots, m_\ell \in S_E(a_\ell)$ (Recall $S_E(r)$ is the exponent set of r).
- Then $m \in S_E(q)$. Therefore m_1 is the sum of a subset of atoms in a factorization of m with respect to M.
- Since M is a FFM, there are finitely many such factorizations.
- Given one such factorization of length ℓ, there are at most 2^ℓ ways to choose m_1.

The FFS Property in $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

If M is a positive monoid that satisfies the FFM property, then $E(M)$ is a FFS.

- Thus, there are finitely many possibilities for m_1, which means there are finitely many ways to choose the exponent set of a_1.
- Since all the nonzero elements of $E(M)$ are more than or equal to one, $a_1 < q$, so all the coefficients of a_1 are less than q.
- This shows that there are finitely many possibilities for a_1.
- Similarly, there are finitely many possibilities for $a_2, \ldots, a_\ell$.
- Thus, there are finitely many factorizations of q with length ℓ.

LFFMs and the $E(M)$ Construction

The FFM, BFM, and ACCP properties ascend to the $E(M)$ construction, but the LFFM property does not.

Proposition (Jiang-Kanungo-Kim, 2023)

Consider a set of rationals

$$S := \left\{ \frac{1}{p_i} \mid i \in \mathbb{N}, i \geq 3 \right\} \cup \left\{ \frac{7}{6p_i} - \frac{1}{p_i^2} \mid i \in \mathbb{N}, i \geq 3 \right\}.$$

Then the monoid M generated by S is an LFFM, but $E(M)$ is not an LFFS.

The elements of S can be written as a decreasing sequence, since for any real number $k > 0$, there are finitely many elements of S greater than k. Thus M is decreasing and is hence an LFFM.

A Counterexample to LFFM Ascending to $E(M)$

Proposition (Jiang-Kanungo-Kim, 2023)

Consider a set of rationals

$$S := \left\{ \frac{1}{p_i} \mid i \in \mathbb{N}, i \geq 3 \right\} \cup \left\{ \frac{7}{6p_i} - \frac{1}{p_i^2} \mid i \in \mathbb{N}, i \geq 3 \right\}.$$

Then the monoid M generated by S is an LFFM, but $E(M)$ is not an LFFS.

Outline of proof:

- We can check $S = \mathcal{A}(M)$ (no element of S divides any other). We also get M is atomic.
- Consider $e + e^{\frac{7}{6}} \in E(M)$.
- This can be factored as $e^{\frac{1}{p_i}} \left(e^{\frac{p_i - 1}{p_i}} + e^{\frac{7}{6} - \frac{1}{p_i}} \right)$ for any $i \geq 3$. We claim both of these factors are atoms.
- Note that $e^{\frac{1}{p_i}}$ is an atom in $E(M)$ since $\frac{1}{p_i}$ is an atom in M.
- We want to prove that $e^{\frac{p_i - 1}{p_i}} + e^{\frac{7}{6} - \frac{1}{p_i}}$ is an atom.

$E(M)$ is Not an LFFS

We will introduce a definition to help reduce our problem.

Definition

Let the number of terms of $x \in E(M)$ be the unique positive integer k such that there is a multiset T of size k of rationals with $\sum_{t \in T} e^t = x$.
Informally: what we get when we replace e with 1.

- Note that the number of terms is multiplicative.
- If we could factor $e^{\frac{p_i - 1}{p_i}} + e^{\frac{7}{6} - \frac{1}{p_i}}$, one factor must have 1 term and one must have 2 terms.
- Thus, it suffices to prove that there is no $m \in M$ dividing both $\frac{p_i - 1}{p_i}$ and $\frac{7}{6} - \frac{1}{p_i}$.

$E(M)$ is Not an LFFS

We want to prove that there is no $m \in M$ dividing both $\frac{p_i-1}{p_i}$ and $\frac{7}{6} - \frac{1}{p_i}$. As an example, we first find factors of $\frac{p_i-1}{p_i}$. Outline of proof:

- If $\frac{7}{6p_j} - \frac{1}{p_j^2}$ is a factor for any $j \geq 3$, we would need at least p_j copies, which is too big.
- Now, if $\frac{1}{p_j}$ divided $\frac{p_i-1}{p_i}$ for $i \neq j$, we would need at least p_j copies (we have ruled out the only other atom with p_j in the demoninator), which is also too big.
- Thus, $\frac{1}{p_i}$ is the only atom dividing $\frac{p_i-1}{p_i}$, and the only factorization is $p_i - 1$ copies of $\frac{1}{p_i}$.

We now want to prove that no factorization of $\frac{7}{6} - \frac{1}{p_i}$ includes $\frac{1}{p_i}$, which is a similar proof so we do not show it here.

Thus, the factorization $e^{\frac{1}{p_i}}\left(e^{\frac{p_i-1}{p_i}} + e^{\frac{7}{6}-\frac{1}{p_i}}\right)$ is comprised of two atoms. There are infinitely many ways to factor $e + e^{\frac{7}{6}}$ into two atoms, so $E(M)$ is not an LFFS.

When LFFM Ascends

We can impose some conditions on the monoid M so that $E(M)$ is indeed an LFFS when M is an LFFM.

Theorem (LFFM Ascent Theorem, Jiang-Kanungo-Kim, 2023)

Let M be an atomic positive LFFM of rational numbers. Assume that there exists an integer N such that, for all $n > N$, there is at most one atom of M with denominator divisible by n in simplest form. Then $E(M)$ is an LFFS.

Proposition (Jiang-Kanungo-Kim, 2023)

If M is Grams' monoid, defined as

$$M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle,$$

then $E(M)$ is an LFFS.

We will prove this by analyzing properties of a general monoid M.

The Structure of Elements of E(M)

Definition

Let two elements $x_1, x_2 \in E(M)$ have the same structure if $\frac{x_2}{x_1} = e^q$ for some rational number q (not necessarily in M).

Definition

Consider some $x \in E(M)$. Let two factorizations z_1, z_2 of x have the same structure if z_1 and z_2 have the same number of atoms, and we can pair up atoms in the two factorizations such that each pair has two atoms with the same structure.

As an example, notice that in the semiring mentioned in the previous proof,

$$e^{\frac{1}{p_i}} \left(e^{\frac{p_i - 1}{p_i}} + e^{\frac{7}{6} - \frac{1}{p_i}} \right)$$

has the same structure across any $i \geq 3$.

Reducing the Problem to One Structure

Consider some $x \in E(M)$ and positive integer ℓ. We claim that factorizations of x of length at most ℓ take on only finitely many distinct structures. Outline of proof:

- The structure of an element of $E(M)$ depends on the pairwise differences between exponents and on the coefficients of linearly independent terms.
- The number of terms of each factor is bounded by the number of terms of x, and so there are finitely many choices for each coefficient.
- The differences between exponents must all also be differences of exponents in x, giving only finitely many possible differences.
- Thus, there is a finite number of ways to choose the structure of each factor.
- Since there are finitely many factors (at most ℓ), there are finitely many distinct structures among factorizations.

Thus, it suffices to show that any given structure produces finitely many factorizations.

Rephrasing

We will rephrase the question to be more convenient.

- Factor out e^q for rational q as large as possible, so all factors in the factorization have smallest term 1 (and are possibly no longer in $E(M)$).
- Let the resulting "factors" with smallest term 1 be $z_1, z_2, \ldots, z_k$ where $k \leq \ell$ is a positive integer.
- Example: in the case of $e^{\frac{1}{p_i}}\left(e^{\frac{p_i-1}{p_i}} + e^{\frac{7}{6}-\frac{1}{p_i}}\right)$, we would get $z_1 = 1$, $z_2 = 1 + e^{\frac{1}{6}}$, and $q = 1$.
- Each factorization is equivalent to a sequence of rational numbers $q_1, q_2, \ldots, q_k$ that sum to q, such that $z_i e^{q_i}$ is an atom in $E(M)$ for any $i \in \{1, 2, \ldots, k\}$.
- Example: in $e^{\frac{1}{p_i}}\left(e^{\frac{p_i-1}{p_i}} + e^{\frac{7}{6}-\frac{1}{p_i}}\right)$, we have $q_1 = \frac{1}{p_i}$ and $q_2 = \frac{p_i-1}{p_i}$.

We can now prove our two results.

First Theorem

Theorem (LFFM Ascent Theorem, Jiang-Kanungo-Kim, 2023)

Let M be an atomic LFFM. Assume that there exists an integer N such that, for all $n > N$, there is at most one atom of M with denominator divisible by n in simplest form. Then $E(M)$ is an LFFS.

As before, let $x \in E(M)$ be the element we are studying, and define $z_1, z_2, \ldots, z_k$ and $q, q_1, \ldots, q_k$ as in the previous section. First, pick a very large n with $n > N$, n greater than the denominator of any exponent in $z_1, z_2, \ldots, z_k$ in simplest form, or the denominator of q in simplest form, and $n > k$. Our main claim is:

Lemma (Jiang-Kanungo-Kim, 2023)

The denominators of $q_1, \ldots, q_k$ in simplest form are not divisible by any primes p with $p > n$.

Proving the Lemma

Lemma (Jiang-Kanungo-Kim, 2023)

The denominators of $q_1, \ldots, q_k$ in simplest form are not divisible by any primes p with $p > n$.

Outline of proof:

- By assumption, only one atom has p in the denominator. Call this atom a.
- We first deal with any of the z_i being equal to 1.
  - If any of $z_1, \ldots, z_k$ are simply 1, we may let them be $z_1, \ldots, z_j$ with $j \leq k$.
  - If $j = k$, then we simply have $x = e^q$, so since q has finitely many factorizations of length at most ℓ, so does x. Thus, we may assume $j < k$.
- Now, let p be a prime greater than n, and assume for the sake of contradiction that p divides the denominator of q_i for some $i \in \{1, \ldots, k\}$.
- If $i \leq j$, then since $p > k$, and only one atom has p in the denominator, if we assign these only to q_i for $i \leq j$, then $q_1 + \cdots + q_k$ still has a p in the denominator, impossible.

Proving the Lemma

Lemma (Jiang-Kanungo-Kim, 2023)

The denominators of $q_1, \ldots, q_k$ in simplest form are not divisible by any primes p with $p > n$.

Outline of proof (continued):

- We can thus assume $i > j$.
- Then, every exponent in $z_i e^{q_i}$ has denominator divisible by p in simplest form.
- Since M is atomic, a must divide every exponent of $z_i e^{q_i}$.
- Then, $z_i e^{q_i}$ is not an atom since it is divisible by e^a, which is a contradiction. This proves the lemma.

Notice that, in the lemma, we only used the assumption that there is one prime with denominator divisible by large enough p for *prime* p, not the full assumption for all positive integers.

First Theorem

We now return to proving the theorem.

Theorem (LFFM Ascent Theorem, Jiang-Kanungo-Kim, 2023)

Let M be an atomic LFFM. Assume that there exists an integer N such that, for all $n > N$, there is at most one atom of M with denominator divisible by n in simplest form. Then $E(M)$ is an LFFS.

Outline of proof:

- Now, we have that a finite number of primes divide the denominators of $q_1, q_2, \ldots, q_k$.
- By assumption, there is only one atom with denominator divisible by a large power of any prime.
- This lets us bound the denominators of $q_1, \ldots, q_k$ with a product of large, but finite, prime powers (since $q_1, \ldots, q_k$ are all in M).
- If we let d be the least common multiple of all possible denominators, then dq_i is an integer for any $i \in \{1, \ldots, k\}$. But all q_i are bounded by q, so we have a finite number of choices, as desired.

Grams' Monoid

We can also use a similar argument for Grams' Monoid:

Proposition (Jiang-Kanungo-Kim, 2023)

If M is Grams' monoid, defined as

$$M := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\rangle,$$

then $E(M)$ is an LFFS.

Let all notation be the same as the last proof.

Grams' Monoid

We present a very brief outline of the proof:

- Note that M satisfies the property that only one atom has denominator divisible by p for large enough *prime* p.
- Thus, we may similarly prove that q_i does not have arbitrarily large primes in its denominator for all $i \in \{1, 2, \ldots, k\}$.
- However, there is an extra nuance: we can have atoms with arbitrarily large primes in its denominator sum to elements of M without arbitrarily large primes in its denominator (such as $\frac{1}{2^n}$ for arbitrarily large n).
- We now add an additional claim: we also cannot have arbitrarily large powers of two in the denominators of any of the q_i. The proof is very similar to the previous lemma.
- Therefore, there are only finitely many choices of atoms dividing the q_i, and thus finitely many choices of the q_i themselves, as desired.

Corollaries of Propositions

Because of the above results, we are able to extend the examples given for monoids to show that LFFS is independent from BFS and ACCP for semirings, as well.

Example of an LFFS that does not satisfy the ACCP:

- Consider the Grams' monoid: $M := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$.
- By our previous proposition, $E(M)$ is an LFFS.
- We know that M does not satisfy the ACCP. Since e^M has the same algebraic structure as M, the fact that M does not satisfy the ACCP guarantees that $E(M)$ does not satisfy the ACCP.

Example of a BFS that is not an LFFS:

- Consider the positive monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$.
- We have previously seen that M is a BFM with $\mathcal{A}(M) = \mathbb{Q} \cap [1, 2)$ and also that M is not an LFFM.
- By our previous proposition that bounded factorization property ascends to $E(M)$, we know that $E(M)$ is a BFS.
- Since e^M has the same algebraic structure as M, the fact that M is not an LFFM guarantees that $E(M)$ is not an LFFS.

Related Open Problems

We define a structure related to the $E(M)$ construction, the monoid algebra.

Definition

Let F be a field of rational numbers. Then $F(M)$ is an integral domain generated by

$$\{fe^m \mid f \in F, m \in M\},$$

called a monoid algebra of M over F.

Intuitively, this is like the $E(M)$ construction, but the coefficients are in some field F instead of the nonnegative integers. Some questions are:

Question

Does the LFFM property ascend from M to $F(M)$ for certain fields F?

Question

Does the FFM property ascend from M to $F(M)$ for certain fields F?

This adds the complication that some coefficients may be negative.

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Thank you!